Optimal Designs for Multi-Response Experiments

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Introduction

Designed experiments play an instrumental role throughout the various stages of product or process development. There are usually multiple performance and quality metrics that are of interest in an experiment, but much of the research in designed experiments is focused on having only one response variable. There is relatively little research done to address multi-response experiments with responses following different distributions.

*Examples of Experiments with Different Response Types

- Missile Warning System:
  - Detection (binary)
  - Timely Notification (continuous)
  - False Alarms (count)
- Air-to-Air Missile:
  - Survival of Target (binary)
  - Miss Distance (continuous)
  - Time to Acquire Target (continuous)

Objectives

- Provide a general solution to finding a test design for multiple responses that follow different distributions
- Identify a method for finding optimal weights when using a weighted design criterion

Weighted Optimality Criterion

Consider a three-response problem with dependent variables following a normal, a binomial, and a Poisson distribution. The algorithm searches for the design that maximizes the weighted optimality criterion φ_W.

$$\varphi_W = w_N\tilde{\varphi}_N + w_B\tilde{\varphi}_B + w_P\tilde{\varphi}_P$$

Table 1: Notation and functions used in algorithm

Response Type	Normal Continuous Data	Binomial Probability of Success	Poisson Count Data
Response Model	$y_i = \boldsymbol{x}_i'\boldsymbol{\beta}_N + \varepsilon_i$ $\varepsilon_i \sim N(0, \sigma^2)$	$y_i = \frac{exp(\boldsymbol{x}_i'\boldsymbol{\beta}_B)}{1 + exp(\boldsymbol{x}_i'\boldsymbol{\beta}_B)}$	$y_i = exp(\boldsymbol{x}_i'\boldsymbol{\beta}_P)$
D-Criterion	$\varphi_N = \lvert\boldsymbol{X}'\boldsymbol{X}\rvert$	$\varphi_B = \int \lvert\boldsymbol{X}'\boldsymbol{V}_B\boldsymbol{X}\rvert f(\boldsymbol{\beta_B})d\boldsymbol{\beta_B}$	$\varphi_P = \int \log\lvert\boldsymbol{X}'\boldsymbol{V}_P\boldsymbol{X}\rvert\, f(\boldsymbol{\beta_P})d\boldsymbol{\beta_P}$
Information Matrix	$\boldsymbol{X}'\boldsymbol{X}$	$\boldsymbol{X}'\boldsymbol{V}_B\boldsymbol{X}$	$\boldsymbol{X}'\boldsymbol{V}_P\boldsymbol{X}$
Weight Matrix		$\boldsymbol{V}_B = \frac{\exp(\boldsymbol{x}_i'\boldsymbol{\beta}_B)}{(1 + \exp(\boldsymbol{x}_i'\boldsymbol{\beta}_B))^2}$	$\boldsymbol{V}_P = \exp(\boldsymbol{x}_i'\boldsymbol{\beta}_P)$
Optimal Design Criterion	$\varphi_{N_{opt}}$	$\varphi_{B_{opt}}$	$\varphi_{P_{opt}}$
Current Design Criterion	$\varphi_{N_{crit}}$	$\varphi_{B_{crit}}$	$\varphi_{P_{crit}}$
Scaled Criterion (Efficiency)	$\tilde{\varphi}_N = \frac{\varphi_{N_{crit}}}{\varphi_{N_{opt}}}$	$\tilde{\varphi}_B = \frac{\varphi_{B_{crit}}}{\varphi_{B_{opt}}}$	$\tilde{\varphi}_P = \frac{\varphi_{P_{crit}}}{\varphi_{P_{opt}}}$

Results

Algorithm is demonstrated for 16 run designs with 2 continuous factors and a main effects plus 2 factor interaction model.

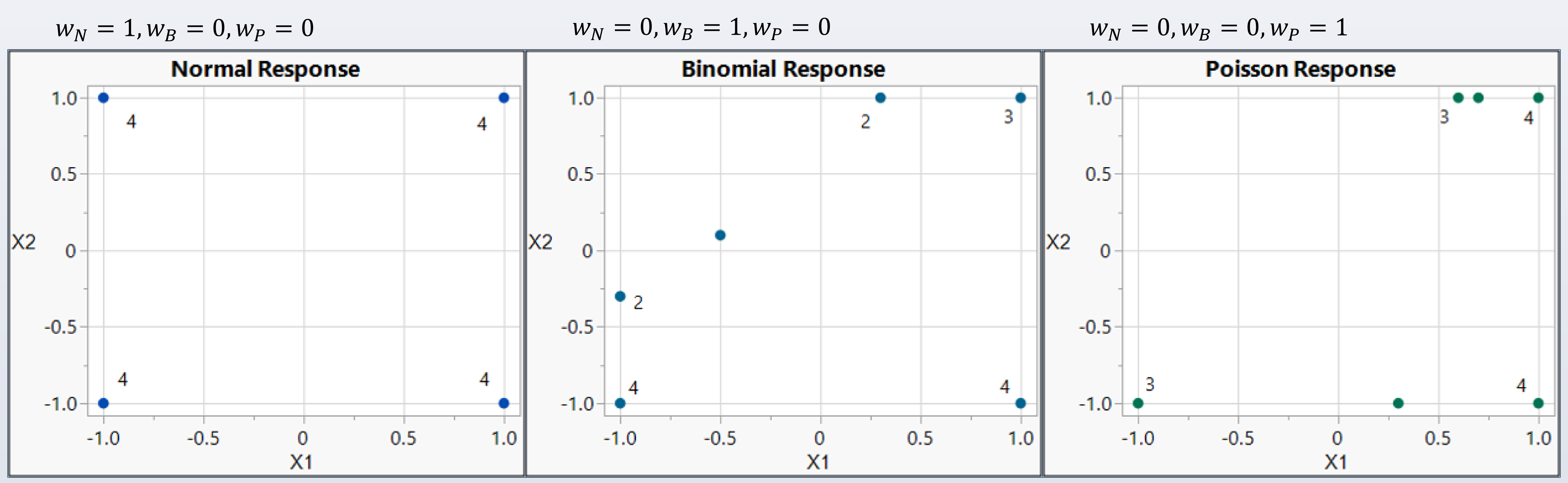

Figure 1: Optimal designs for individual responses; number of replicate design points are labeled.

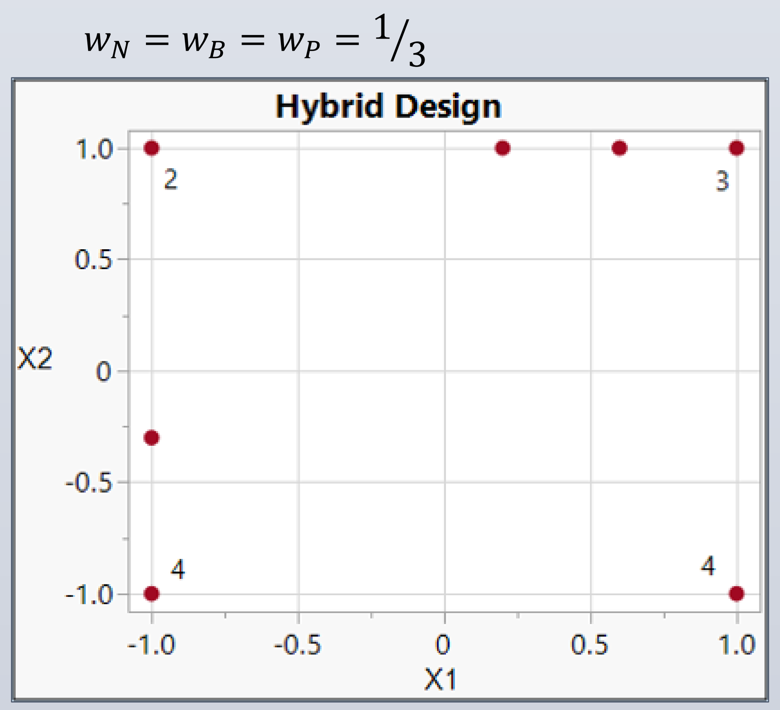

Figure 2

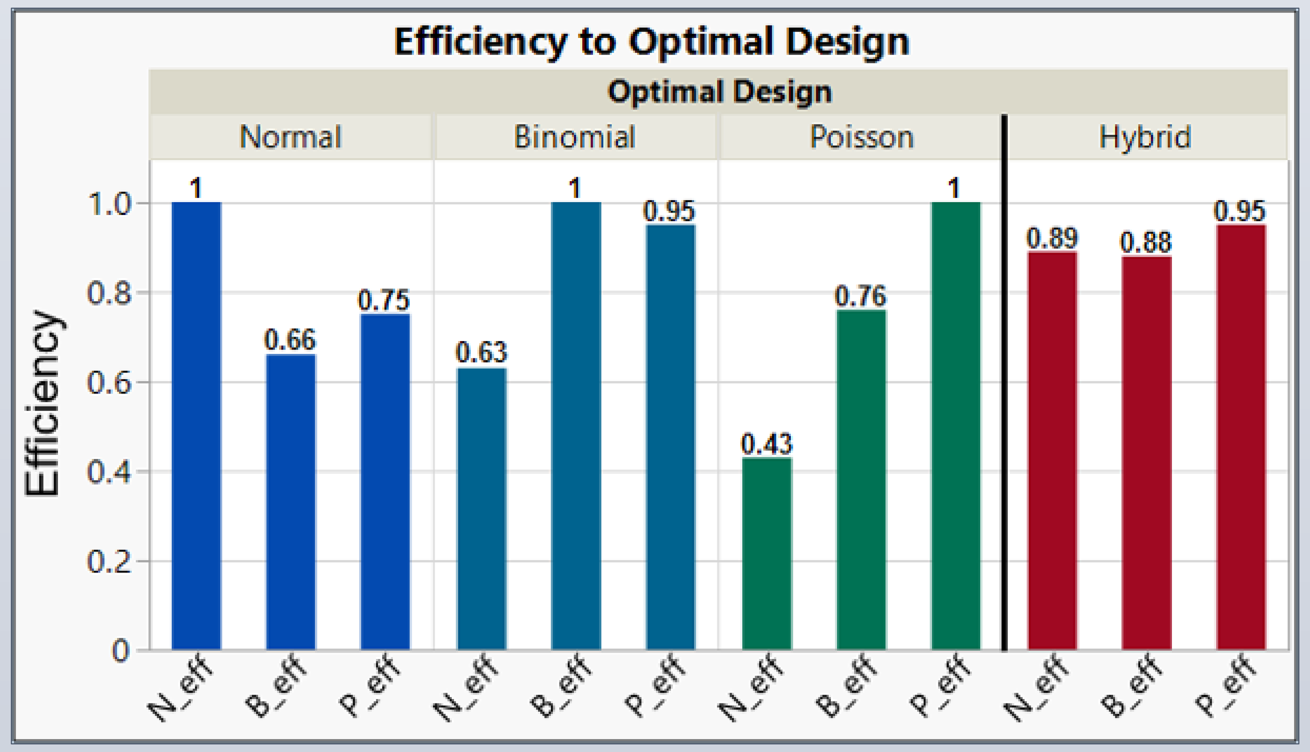

Figure 3

To create these hybrid designs, the user must specify values for w_N, w_B, and w_P. However, the weights in a weighted criterion do not necessarily translate to the specified prioritization. Therefore, designs with various sets of weights should be considered to truly identify the best weighted design for all three response types. The proposed method to vary the weights systematically is to regard the problem as a mixture experiment by requiring $w_N + w_B + w_P = 1$.

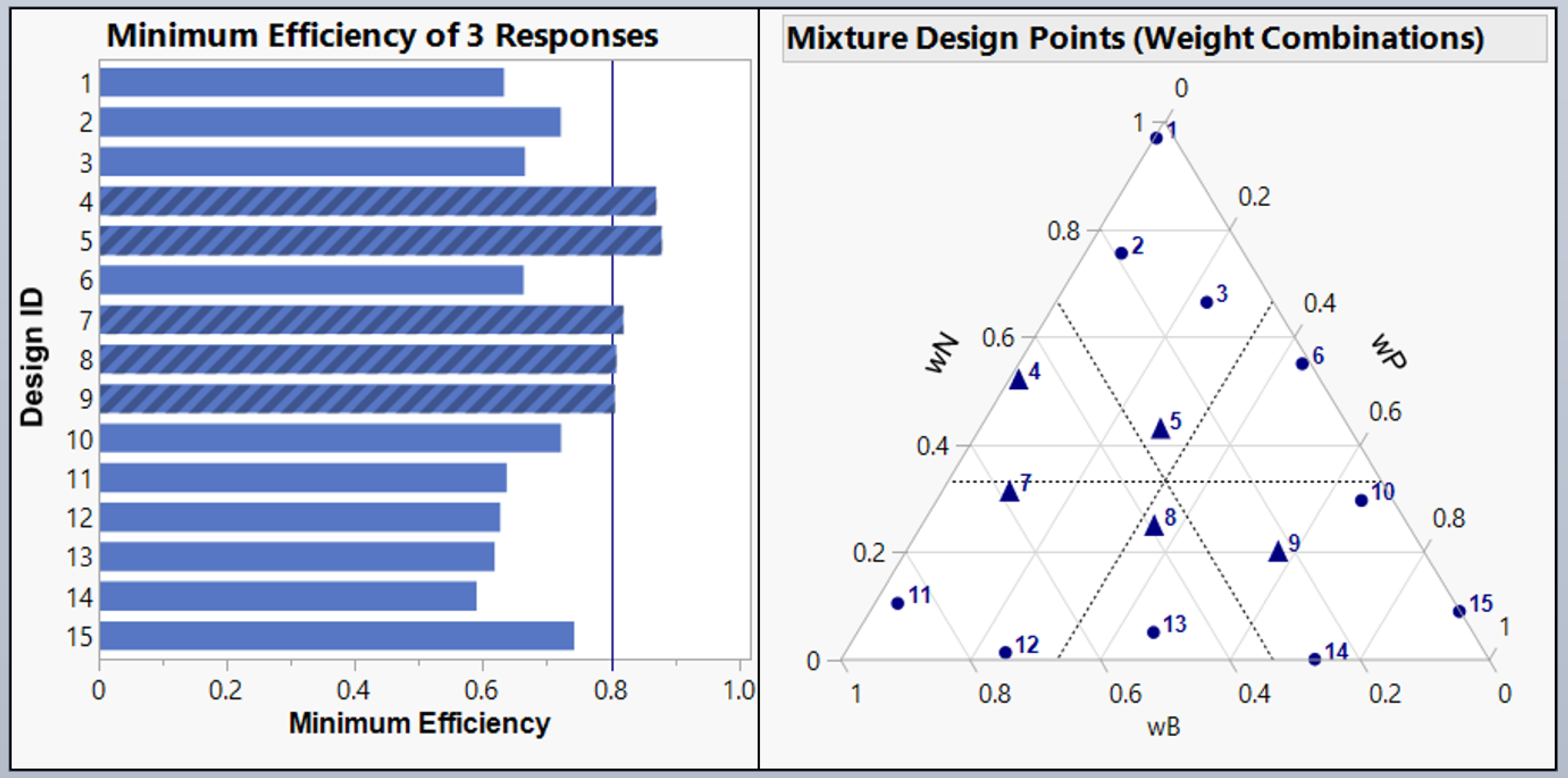

Figure 4

Interpretation of Results

Figure 2 is a hybrid optimal design using equal weights $w_N = w_B = w_P = 1/3$ to balance the priority of each response equally.

Figure 3 compares the hybrid design to using one of the individual optimal designs instead. Each bar represents the efficiency calculations from the bottom row of Table 1. These values represent ratios of the current design criterion to its corresponding optimal design criterion for a given set of inputs. The hybrid design is more than 80% efficient to all three response types.

The numbered points shown on the ternary plot in Figure 4 are the values of 15 sets of weights for the three-response problem generated from a space filling mixture design. The reference line drawn at 0.8 on the bar chart in Figure 4 identifies the sets of weights that produce hybrid designs at least 80% efficient for all response types. This gives the experimenter a subset of 5 design choices (Figure 5):

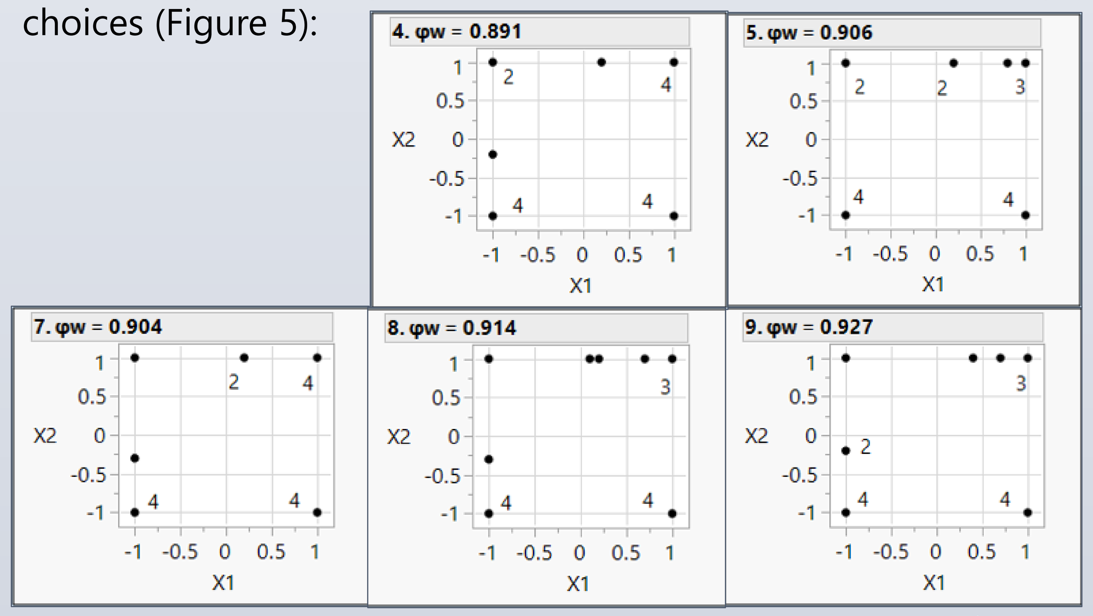

Figure 5

Conclusions

The weighted optimality criterion and efficiency values are the metrics used to determine the best designs. The individual optimality criteria $\varphi_{N_{opt}}$, $\varphi_{B_{opt}}$, and $\varphi_{P_{opt}}$ serve as a point of reference. They are the best values for the given set of design inputs. The efficiency values show that the hybrid design is superior when resources allow for only one experiment to be run. Using any of the individual optimal designs causes the models for the other two response types to suffer. Future work will also include using power to assess the optimal designs.

The user must specify prior distributions on the parameters for the binomial and Poisson models. The distributions used in this example are included in Table 2. Further research will include a formal sensitivity analysis on the prior distributions and their effect on the optimal designs.

Table 2

	Prior Distribution	
Parameter	Binomial	Poisson
β_0	$N(0.7, 0.3^2)$	$N(1.75, 0.625^2)$
β_1	$N(-2.35, 0.825^2)$	$N(3.25, 1.375^2)$
β_2	$N(4.25, 1.875^2)$	$N(-1.875, 0.8125^2)$
β_{12}	$N(-1.75, 0.625^2)$	$N(2.375, 0.5625^2)$

Mixture designs offer an efficient way to search for optimal weights when using a weighted optimality criterion. If desired, sequential designs can be done to perform a local search for optimal weights by implementing linear constraints on the design space.

*Examples courtesy of Jim Simpson

Algorithm was implemented using JMP®, Version *Pro 16.0.0*