Profile Monitoring via Eigenvector Perturbation

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Definition (Profile)

A profile is a noise perturbed functional relationship such that

$$y_i^t = f(\boldsymbol{x}_i^t) + \epsilon_i^t, \quad i \in \{1, \dots, n\} = [n], \quad t = 1, 2, \dots$$

where $y_i^t \in \mathbb{R}$ is a noisy response and $\boldsymbol{x}_i^t \in \mathbb{R}^d$ are known predictors.

Change Point Framework

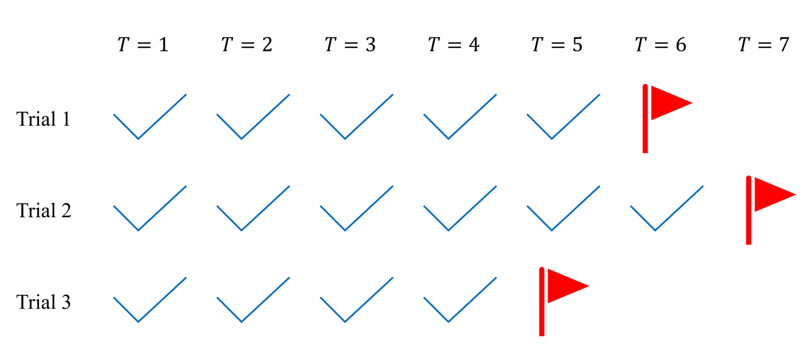
For fixed $\tau < T$, for all $i \in [n], t \in \{1-m, \dots T\}$,

$$y_i^t = \begin{cases} f(\boldsymbol{x}_i^t) + \epsilon_i^t, & t \le \tau \quad \text{"in-control"} \\ h(\boldsymbol{x}_i^t) + \epsilon_i^t, & t > \tau \quad \text{"out-of-control"} \end{cases} \qquad \begin{array}{l} H_0: f^0 = f^1 = \dots = f^T \\ H_a: f^0 = f^1 = \dots = f^\tau \neq f^{\tau+1} = \dots = f^T \end{array}$$

where $t \le 0$ denotes historical, known IC profiles.

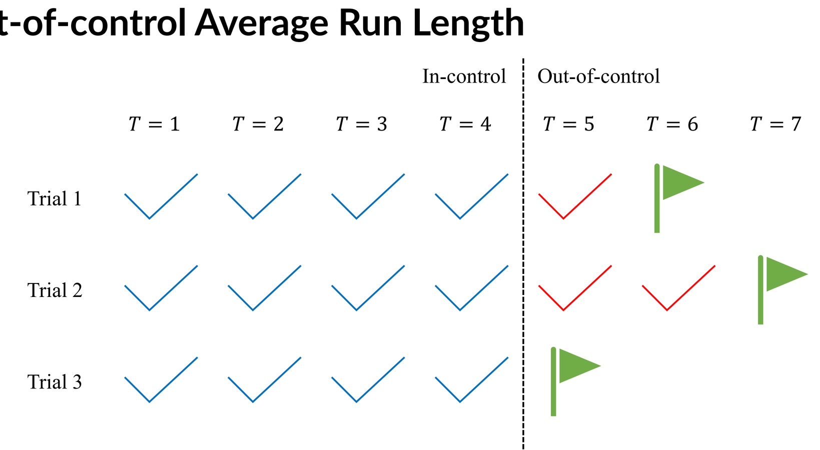
Performance Metrics

In-control Average Run Length



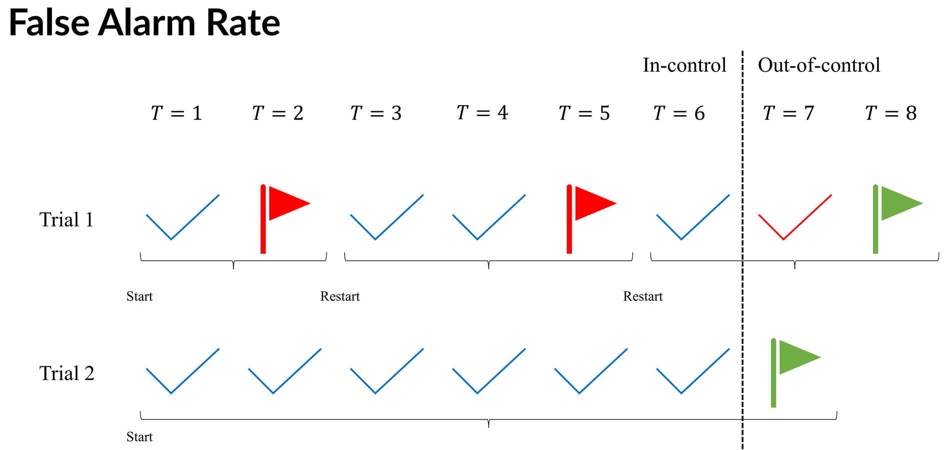
$$ARL_0 = \frac{6+7+5}{3} = 6$$

Out-of-control Average Run Length



$$ARL_1 = \frac{2+3+1}{3} = 2$$

False Alarm Rate



$$\text{FAR} = \frac{\text{\# of restarts due to FA}}{\text{\# of starts}} = \frac{2}{4} = 0.5$$

Goals

Model Assumptions

- ✓ Nonlinear f
- ✓ Nonparametric
- ✓ Multivariate predictor
- ✗ Predictors can change during monitoring

Performance

- ✓ Computationally fast
- ✓ Low FAR even at large τ
- ✓ Fast detection (low ARL_1)

Eigenvector Perturbation

General Setting

Define $\boldsymbol{M}$ such that

$$\boldsymbol{M} = \widetilde{\boldsymbol{M}} + \boldsymbol{E}$$

where $\widetilde{\boldsymbol{M}} \in \mathbb{R}^{w \times w}$ is fixed and $\boldsymbol{E}$ is random.

Matrix	$\boldsymbol{M}$	$\widetilde{\boldsymbol{M}}$
Eigenvectors	$\boldsymbol{v}_1, \dots, \boldsymbol{v}_w$	$\tilde{\boldsymbol{v}}_1, \dots, \tilde{\boldsymbol{v}}_w$

Our Setting

Define $\boldsymbol{E}$ such that

$$\boldsymbol{R} = \mathbb{E}[\boldsymbol{R}] + \boldsymbol{E}$$

where $\boldsymbol{R}$ is a sample correlation matrix of responses $\{\boldsymbol{y}^{T-w+1}, \dots, \boldsymbol{y}^T\}$

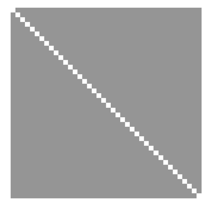
Matrix	$\boldsymbol{R}$	$\mathbb{E}[\boldsymbol{R}]$
Leading eigenvector	$\boldsymbol{v}$	$\tilde{\boldsymbol{v}}$

Eigenvector Perturbation Question

"Under certain assumptions on $\widetilde{\boldsymbol{M}}$ and $\boldsymbol{E}$, how 'far' can $\boldsymbol{v}_j$ be from $\tilde{\boldsymbol{v}}_j$?"

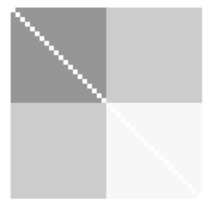
Main Idea

All w Profiles are In-control

$\mathbb{E}[\boldsymbol{R}] =$ 

Eigenvector	Eigenvalue
$\tilde{\boldsymbol{v}} = \frac{1}{\sqrt{w}}\boldsymbol{1}$	$1 + \gamma_1(w-1)$
$\boldsymbol{e}_1 - \boldsymbol{e}_j, \; j \in [w]$	$1 - \gamma_1$

Mix of In-control and Out-of-Control

$\mathbb{E}[\boldsymbol{R}] =$ 

$$\tilde{\boldsymbol{v}} \propto \xi \begin{bmatrix} \boldsymbol{1} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix}$$

where ξ is the root of a certain quadratic.

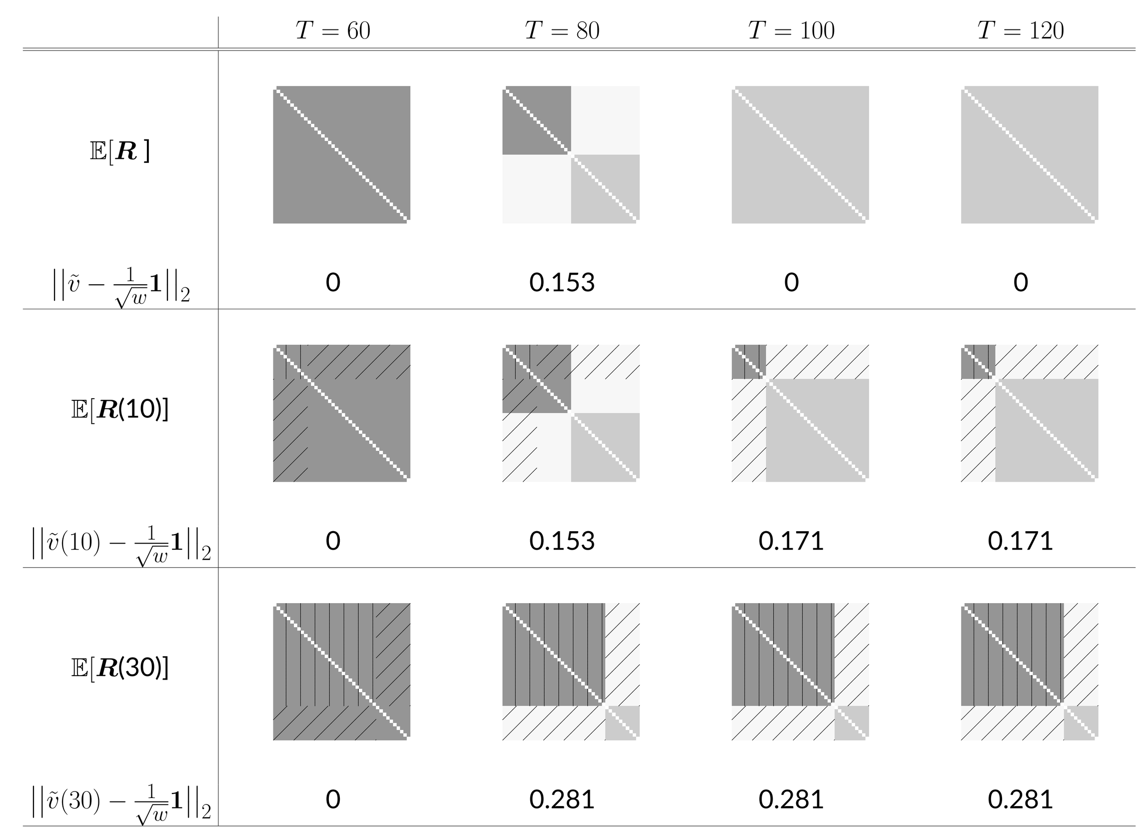
Monitoring Statistic

As $\boldsymbol{v} \approx \tilde{\boldsymbol{v}}$, $||\boldsymbol{v} - \frac{1}{\sqrt{w}}\boldsymbol{1}||_2$ is *small* under IC conditions and *large* under a mix of IC & OOC conditions.

Fixing a Windowing Problem

Denote $\boldsymbol{R}(k)$ as $\boldsymbol{R}$ but with the oldest k profiles replaced by a sample of k historical profiles.

Example: $w = 40$, $\tau = 60$, and $\tilde{v}(k)$ is the leading eigenvector of $\mathbb{E}[\boldsymbol{R}(k)]$.



Modified Monitoring Statistic

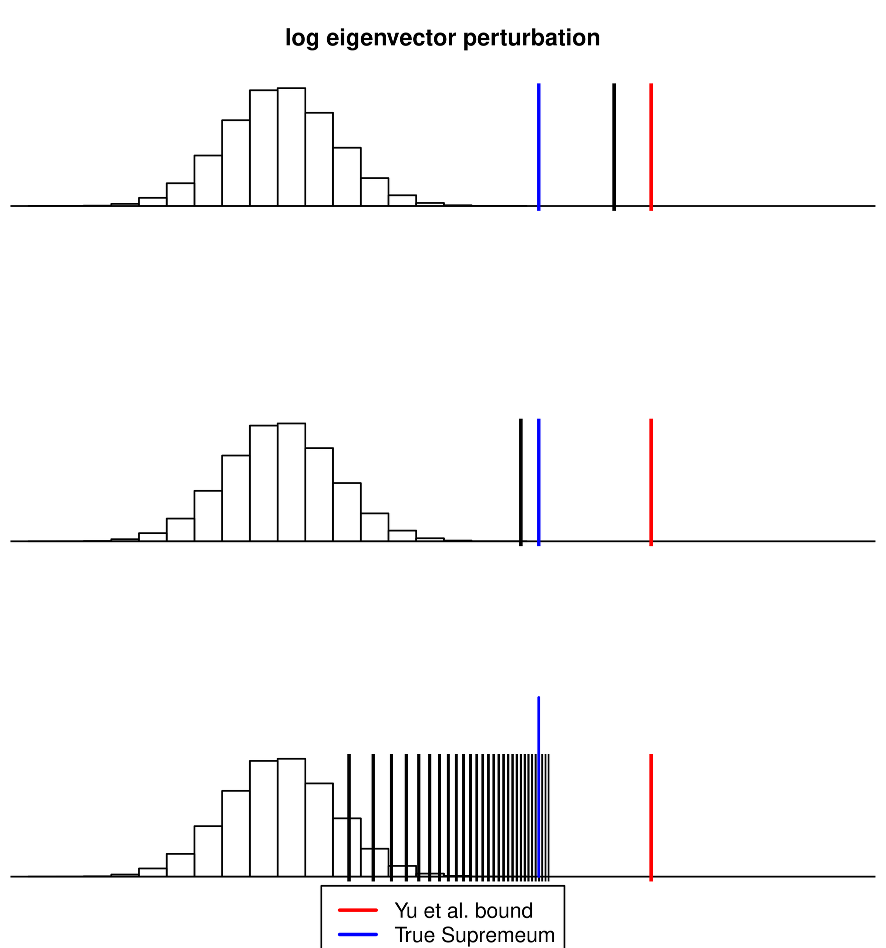
Consider a set of L almost evenly spaced values in $\{0, \dots, w\}$,

$$K = \left\{1, \left\lfloor \frac{w}{L} \right\rfloor, 2\left\lfloor \frac{w}{L} \right\rfloor, \dots, (L-2)\left\lfloor \frac{w}{L} \right\rfloor, w-1 \right\}.$$

The monitoring statistic with the new windowing procedure is

$$\max_{k \in K} ||\boldsymbol{v}(k) - \frac{1}{\sqrt{w}}\boldsymbol{1}||_2.$$

Control Limit via a Quantile Trick



$ARL_0 = \infty$

$ARL_0 < \infty$

$\boldsymbol{e}_j^\top(\boldsymbol{v} - \tilde{\boldsymbol{v}}) \to N(0, \cdot)$ Cape et al.[1]

$UCL = (1-c)$th normal quantile, c small (e.g., 10^{-14})

Simulations Study: Wins against competitors

In-control profile	m	ARL_0^*	Finished by $T = 10^7$	Lower bound on ARL_0
quadratic	20	3947093	35	7881483
linear	20	2553138	37	7244662
quadratic	40	4362645	29	8365168
linear	40	2815431	52	7068576

Table 1. Lower bounds on the ARL_0 using in-control profiles from [2]

Method	ARL_0	τ	Range of observed FAR	Range of observed ARL_1	Fast Calibration?	Unknown f allowable?
Li et al.[3]	200	30	$(0.17, 0.37)$	$(4.24, 4.86)$	No	No
	370	30	$(0.07, 0.34)$	$(4.37, 5.39)$	No	No
Iguchi et al.[2]	200	30	$(0.07, 0.25)$	$(1, 2.44)$	No	No
	370	30	$(0.01, 0.11)$	$(1, 2.75)$	No	No
Eigenvector Perturbation	200	30	$(0.17, 0.69)$	$(1, 1)$	No	No
	370	30	$(0.01, 0.55)$	$(1, 1)$	No	No
	$> 7 \times 10^6$	30	$(0, 0)$	$(1, 1)$	Yes	Yes
	$> 7 \times 10^6$	10^4	$(0, 0.01)$	$(1, 1)$	Yes	Yes

Table 2. Performance comparison using a subset of the profile combinations from [2]

Simulations Study: When does performance degrade?

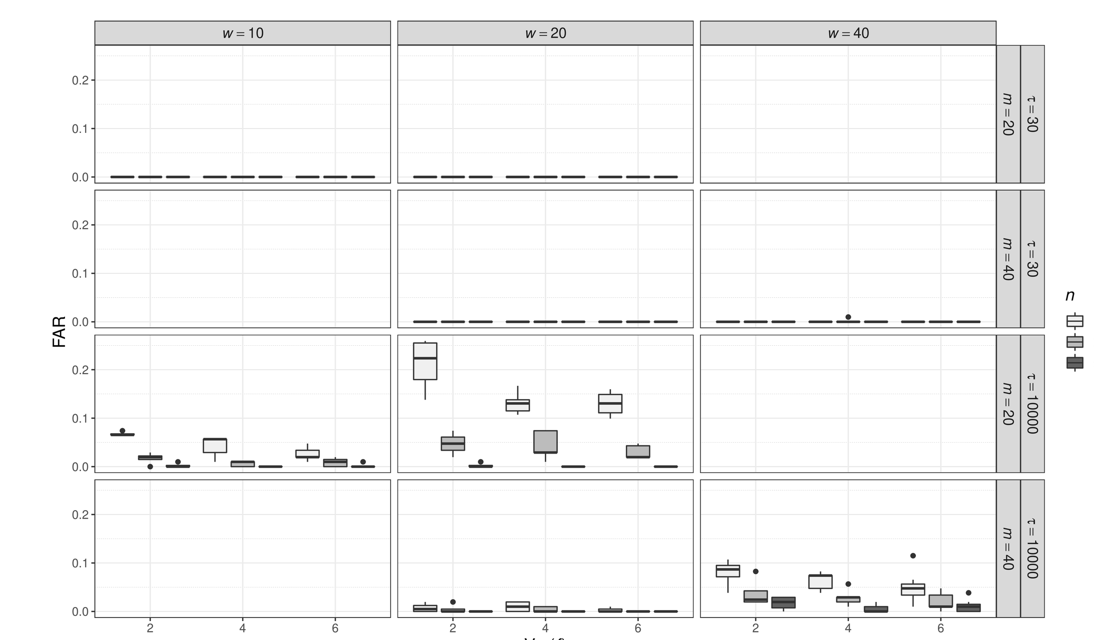
Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{25 \times 25}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^{25}$.

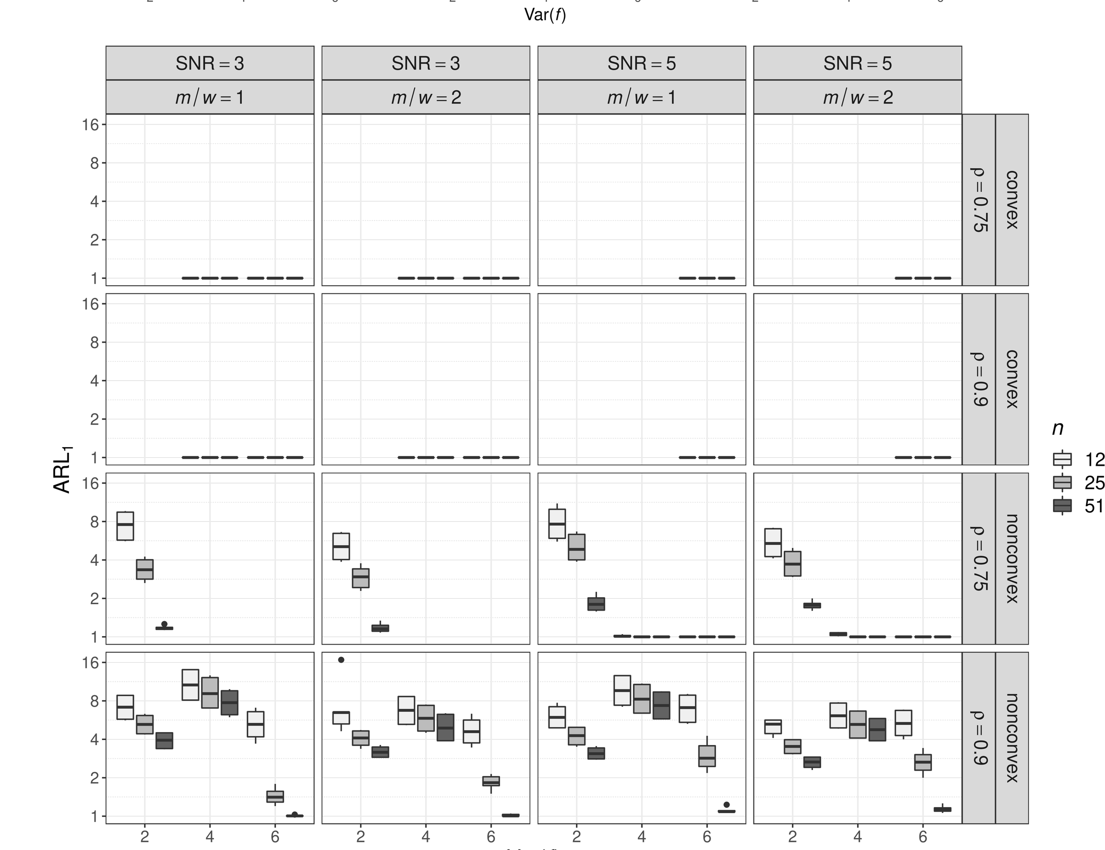
In-control: $f(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{x} + \boldsymbol{a}^\top \boldsymbol{x}$

Out-of-control direction: $g(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{B} \boldsymbol{x} + \boldsymbol{b}^\top \boldsymbol{x}$

Out-of-control: $h(\boldsymbol{x}) = \nu f(\boldsymbol{x}) + (1-\nu) g(\boldsymbol{x})$

Factor	Level
τ	0, 30 , 10^4
n	128, 256, 512
m	20, 40
m/w	1, 2
SNR	3, 5
$\text{Var}(f)$	2,4,6
$\rho(f, h)$	0.75, 0.9
convexity	$\nu \in (0, 1), \nu > 1$





References

[1] J Cape, M Tang, and C E Priebe. Signal-plus-noise matrix models: eigenvector deviations and fluctuations. *Biometrika*, 106(1):243–250, 2019.

[2] Takayuki Iguchi, Andrés F. Barrientos, Eric Chicken, and Debajyoti Sinha. Nonlinear profile monitoring with single index models. *Quality and Reliability Engineering International*, 37(7):3004–3017, 2021.

[3] Chung I. Li, Jeh Nan Pan, and Chun Han Liao. Monitoring nonlinear profile data using support vector regression method. *Quality and Reliability Engineering International*, 35(1):127–135, 2019.

[4] Yi Yu, Tengyao Wang, and Richard J. Samworth. A useful variant of the Davis–Kahan theorem for statisticians. *Biometrika*, 102(2):315–323, may 2014.