FAILURE DISTRIBUTIONS FOR PARALLEL DEPENDENT IDENTICAL WEIBULL COMPONENTS

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Applied Motivation

- Turnbuckles on an aircraft are used to hold objects securely in place during flight
- When turnbuckles fail, the objects could fall, causing destruction of the object as well as death or injury to ground personnel
- For an aircraft, there is reason to believe that the turnbuckle reliability rate, the rate at which we expect turnbuckles to survive to a certain time point, is lower than the level required by the aircraft standards
- For our setup, there are two turnbuckles attached to one pylon on the aircraft; the pylon is a suspension device that connects the frame of the aircraft to an engine, fuel tank, or other cargo
- When the first turnbuckle fails, a greater force is exerted on the second turnbuckle causing the distribution of the failure time to change and the risk of an object falling to increase

Assumptions

1. According to the system experts, it can be assumed that the failure times for each turnbuckle follow a Weibull distribution
2. Pairing the two turnbuckles together creates a stochastic dependence between them
   - A failure of one turnbuckle in the pair will ultimately change the failure time distribution for the remaining turnbuckle
   - In order to determine this failure rate, we consider the parallel structure of the components as well as the dependence between the components
3. The Weibull distributions for the components maintain the same shape parameter

Purpose

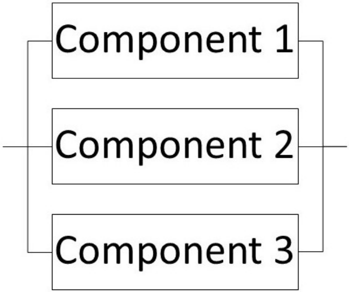
1. Derive cumulative distribution function (CDF) and survival function for a two component system to create a maintenance plan
2. Extend derivation to failure distributions for two failures in n components, three failures in three components, and k failures in n components (not covered in depth)
3. Apply the distributions from the two component system to a simulated data set to predict the survival rate of the pylons

Why is This Important

This research determines the failure distribution for any number of components **without needing to know the correlation between the components**, instead it assumes changes in failure rate associated with the number of surviving components.

Parallel Systems

In a simple three component system, only one component needs to be operational for the system to be operational



Parallel Systems

System failure time and system reliability for these three independent components operating in parallel is then

$$F_{sys}(x) = F_1(x)F_2(x)F_3(x) \quad (1)$$

$$R_{sys}(x) = 1 - F_{sys}(x) \quad (2)$$

Similarly for a system of n independent components operating in parallel

$$F_{sys}(x) = \prod_{i=1}^{n} F_i(x) \quad (3)$$

$$R_{sys}(x) = 1 - \prod_{i=1}^{n} F_i(x) \quad (4)$$

k-out-of-n Systems

- A certain number of components, k, must function in order for the system to work
- Less reliable than parallel system, as it is only partially redundant
- System failure time and system reliability are

$$F_{sys}(x) = 1 - R_{sys}(x) \tag{5}$$

$$R_{sys}(x) = \sum_{j=k}^{n} \binom{n}{i} F_i(x)^j (1 - F_i(x))^{n-j} \tag{6}$$

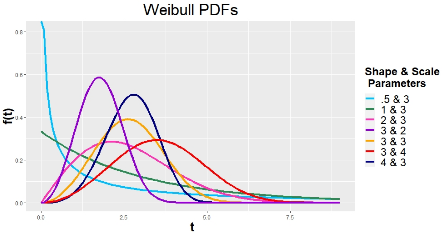
Weibull Distribution

$$F_{Weib}(x|\gamma,\lambda) = 1 - e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}$$

$$f_{Weib}(x|\gamma,\lambda) = \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1} e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}$$

$$R_{Weib}(x|\gamma,\lambda) = e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}$$

$$h_{Weib}(x|\gamma,\lambda) = \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1}$$



Setup

Assumptions:

- Turnbuckles independently tested and follow Weibull distribution
- Two turnbuckles, A and B have the same distribution when operating in parallel
- Failure times, X_A and X_B have marginal distribution $Weibull(\gamma, \lambda)$ while both are operational
- A and B are dependent such that the failure of one component increases the strain on the other through an increased rate of failure

The CDFs of X_A and X_B are given by

$$F_{X_A}(x) = F_{X_B}(x) = 1 - e^{-(\frac{x}{\lambda})^\gamma} \tag{7}$$

for $x \geq 0$, $\gamma > 0$, and $\lambda > 0$

Minimum of Weibull Components

- Let M represent the failure time for the first component to fail
- Then M is the first order statistic for set $\{X_A, X_B\}$
- Since the assumed dependence only affects the second component and $X_A,\ X_B$ are assumed identically distributed, the typical equation holds for the first order statistic, and the CDF of M is

$$\begin{aligned} F_M(x) &= P(X_{(1)} < x) \\ &= 1 - e^{-2(\frac{x}{\lambda})^\gamma} \end{aligned} \tag{8}$$

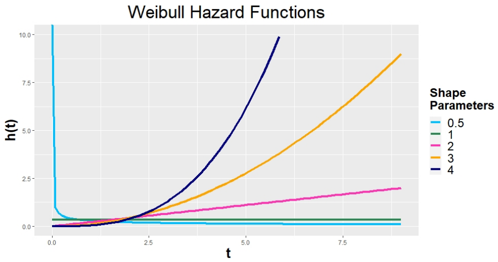
The first turnbuckle failure will follow distribution

$$M \sim Weibull\left(\gamma, \frac{\lambda}{2^{\frac{1}{\gamma}}}\right) \tag{9}$$

Increased Failure Rate

- After the first failure, greater strain is placed on the remaining component
- By assumption, the second component that is leftover, denoted L, will fail with an accelerated rate
- Holding γ constant, we denote θ to be the new scale parameter for L

$$\frac{\gamma}{\theta}\left(\frac{t}{\theta}\right)^{\gamma-1} > \frac{\gamma}{\lambda}\left(\frac{t}{\lambda}\right)^{\gamma-1}$$
$$\theta < \lambda$$



Conditional & Joint Distributions

The failure distribution of L conditioned on M with consideration for accumulated degradation at scale λ up to time M is

$$f_{L|M}(l|m) = \frac{\gamma}{\theta}\left(\frac{l}{\theta}\right)^{\gamma-1} e^{-\left(\frac{l}{\theta}\right)^{\gamma}} e^{\left(\frac{m}{\lambda}\right)^{\gamma}} I_{(0,\infty)}(\gamma, m, \lambda) I_{(0,\lambda)}(\theta) I_{\left(\frac{\theta}{\lambda}m,\infty\right)}(l) \tag{10}$$

The joint distribution of L and M is

$$\begin{aligned} f_{L,M}(l,m) &= f_M(m) f_{L|M}(l|m) \\ &= \frac{\gamma 2^{\frac{1}{\gamma}}}{\lambda}\left(\frac{m 2^{\frac{1}{\gamma}}}{\lambda}\right)^{\gamma-1} \frac{\gamma}{\theta}\left(\frac{l}{\theta}\right)^{\gamma-1} e^{-\left(\frac{l}{\theta}\right)^{\gamma}} e^{-\left(\frac{m}{\lambda}\right)^{\gamma}} I(\gamma,\theta,l,m,\lambda) \end{aligned} \tag{11}$$

with the identity function

$$I(\gamma,\theta,l,m,\lambda) = I_{(0,\infty)}(\gamma) I_{(0,\lambda)}(\theta) I_{\left(\frac{\theta}{\lambda}m,\infty\right)}(l) I_{(0,\infty)}(m) I_{(0,\infty)}(\lambda)$$

Shifted Failure Time

- L is the time to failure of a single component supporting the system alone for the whole time, conditioned to not fail prior to $\frac{\theta}{\lambda}M$
- Note: This is NOT the failure time for the second component
- It is necessary to account for the portion of the life-cycle of the second component operating with failure rate λ
- This failure rate does not take effect until after M has failed
- Let T be the failure time, given by relation

$$T = L + M - \frac{\theta}{\lambda}M = L + \left(1 - \frac{\theta}{\lambda}\right)M \tag{12}$$

Joint Distribution

- Let $\beta = \frac{\theta}{\lambda}$
- $P(T \leq t)$, relies on the dependence defined in the joint distribution of L and M given in equation (11)
- Notice that $0 < \beta < 1$, which logically follows as β is a ratio of the positive failure rates and $\theta < \lambda$

$$f_{T,M}(t,m) = \gamma^2 \left(\frac{2^{\frac{1}{\gamma}}}{\lambda\theta} \right)^{\gamma} (t + (\beta - 1)m)^{\gamma-1} \, m^{\gamma-1} e^{-\left(\frac{t+(\beta-1)m}{\theta}\right)^{\gamma}} e^{-\left(\frac{m}{\lambda}\right)^{\gamma}} I_{(0,\infty)}(\gamma) I_{(0,\lambda)}(\theta) I_{(0,\infty)}(\lambda) I_{(0,t)}(m) I_{(0,\infty)}(t) \quad (13)$$

Distribution of T

Marginalizing over M yields

$$f_T(t) = \gamma^2 \left(\frac{2^{\frac{1}{\gamma}}}{\lambda\theta} \right)^{\gamma} \int_0^t (t + (\beta - 1)m)^{\gamma-1} m^{\gamma-1} e^{-\left(\frac{t+(\beta-1)m}{\theta}\right)^{\gamma}} e^{-\left(\frac{m}{\lambda}\right)^{\gamma}} dm$$
$$I(\gamma, \theta, \lambda, m, t) \tag{14}$$

- Equation (14) has a closed form solution for the special case $\gamma = 1$, when the failure times are assumed distributed Exponential, resulting in a negative mixture of Weibull distributions
- If $\gamma \neq 1$, there is no closed form solution that is not an approximation

Approximation Limitations

- While there is no closed form solution to equation (14) for general $\gamma > 0$, the remaining integral of equation (14) can be approximated using Taylor series expansions

$$f_T(t) = \frac{2\gamma^2\lambda}{t\theta}\left(\frac{t}{\lambda}\right)^{2\gamma}\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{t}{\lambda}\right)^{j\gamma}}{j!}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{t}{\lambda}\right)^{i\gamma}}{i!}$$
$$\sum_{n=0}^{\infty}\left(\frac{\lambda-\theta}{\theta}\right)^n\frac{\binom{\gamma+i\gamma-1}{n}}{(j\gamma+\gamma)\binom{n+j\gamma+\gamma}{n}}I(\gamma,\theta,\lambda,t) \qquad (15)$$

- The form cannot be further collapsed due to the remaining binomial coefficient's dependence on i, j, and n
- The approximation can be validated against the case $\gamma = 1$ under the assumption $0^0 = 1$

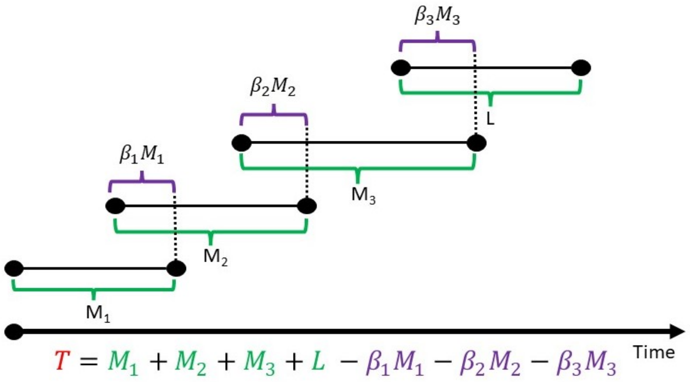
System Failure Time

- Let T_k be the system failure times for a system that has experienced k failures out of n components
- This failure is dependent upon the previous $k-1$ failures denoted M_i where i goes from 1 to $k-1$
- Each failure also has its own associated scale parameter, λ_i, based on the number or components that were operating during that period
- Recall $\beta_i = \frac{\lambda_{i+1}}{\lambda_i}$

$$T_k = M_k + \sum_{i=1}^{k-1} (1 - \beta_i) M_i \tag{16}$$

Visualizing System Lifetime

Equation (16) adds the lifetime of the final remaining component to the lifetimes of the previous component lifetimes scaled for the new time space



Distribution of T

- The transform to T_k follows the same procedure as before using the relation $L = M_k$ where k is the number of failures of interest
- The bounds of integration are defined using the relations between the component failure times: $\beta_1 M_1 \leq M_2$, $\beta_2 M_2 \leq M_3$, ..., $\beta_{k-1} M_{k-1} \leq L$
- The upper bounds can similarly be found for the marginalized variables through the relation defined in equation (16)

$$f_T(t) = \int_0^t \int_{\beta_1 m_1}^{t-(1-\beta_1)m_1} \dots \int_{\beta_{k-1} m_{k-2}}^{t-\sum_{i=1}^{k-2}(1-\beta_i)m_i} \frac{k!\gamma}{\lambda_n} \left(\frac{(t - \sum_{i=1}^{k-1}(1-\beta_i)m_i)}{\lambda_k} \right)^{\gamma-1} e^{-\left(\frac{(t-\sum_{i=1}^{k-1}(1-\beta_i)m_i)}{\lambda_k}\right)^{\gamma}} \prod_{i=1}^{k-1} \frac{\gamma^{k-1}}{\lambda_i} \left(\frac{m_i}{\lambda_i}\right)^{\gamma-1} e^{-(\frac{m_i}{\lambda_i})^{\gamma}} I(\gamma, \lambda_1, \lambda_2, ..., \lambda_n, m_1, m_2, ..., m_{k-1}, l) dm_{k-1} dm_{k-2} \dots dm_1 \quad (17)$$

Application

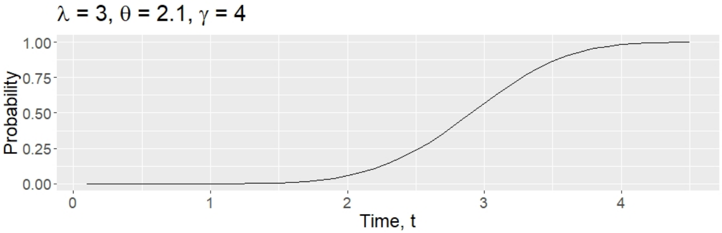
Determining a maintenance plan requires both the CDF and survival function for our two component system.
The CDF (using the summation approximation given in equation (15)) is:

$$F_T(t) = \frac{2\gamma^2\lambda}{\theta}\left(\frac{t}{\lambda}\right)^{2\gamma}\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{t}{\lambda}\right)^{j\gamma}}{j!}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{t}{\lambda}\right)^{i\gamma}}{i!} \sum_{n=0}^{\infty}\left(\frac{\lambda-\theta}{\theta}\right)^n\frac{\binom{\gamma+i\gamma-1}{n}}{(j\gamma+\gamma)\binom{n+j\gamma+\gamma}{n}(2\gamma+j\gamma+i\gamma)}I(\gamma,\theta,\lambda,t) \quad (18)$$

The reliability/survival function is approximated by:

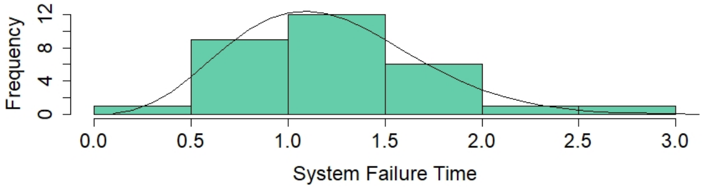
$$R(t) = 1-\left(\frac{2\gamma^2\lambda}{\theta}\left(\frac{t}{\lambda}\right)^{2\gamma}\sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{t}{\lambda}\right)^{j\gamma}}{j!}\sum_{i=0}^{\infty}\frac{(-1)^i\left(\frac{t}{\lambda}\right)^{i\gamma}}{i!} \sum_{n=0}^{\infty}\left(\frac{\lambda-\theta}{\theta}\right)^n\frac{\binom{\gamma+i\gamma-1}{n}}{(j\gamma+\gamma)\binom{n+j\gamma+\gamma}{n}(2\gamma+j\gamma+i\gamma)}\right)I(\gamma,\theta,\lambda,t) \quad (19)$$

Maintenance Plan



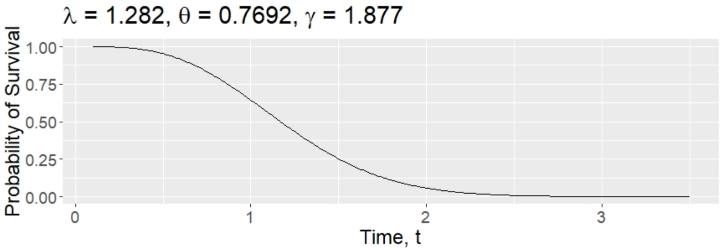
- A plot of the CDF allows system experts to determine if the turnbuckles meet current aircraft standards and when (in some time scale) to perform maintenance on the turnbuckles so pylons will not experience a failure
- To have a 20% chance of system failure with the given parameters, the system should undergo maintenance at 2.43 time units

Empirical System Failure Times



- Estimations of parameter values can be made based on empirical data
- In the example, the failure scale of the leftover component to be about 60% of the original rate, implying $\beta = 0.6$ and $\theta = 0.7692$
- The figure shows the simulated empirical data for failure times of the overall system (based on the specified parameters) overlaid with the PDF for the approximation

Survival Curve



- A survival curve can assist in determining a maintenance schedule
- For a 50% chance of survival, the maintenance action should be taken at no less than 1.17 time units
- For no less than an 80% chance of survival, maintenance should be performed no later than 0.80 time units

Conclusions

- For a two component system, the distribution of the minimum of correlated Weibull components is used to derive both an exact and Taylor series approximation for the PDF
- The pattern can be extended to find the form of the PDF for two failures in n components, three failures in three components, and k failures in n components
- For all of the equations, the dependence between the components is allowed to remain unknown
- When the Taylor series holds, the derivations can be used to construct a survival curve and the desired maintenance schedule