Debunking Stress Rupture Theories Using Weibull Regression Plots

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NASA Strand and Vessel Testing

• NASA’s Engineering Safety Center (NESC) project to assess safety of Composite Overwrapped Pressure Vessels (COPVs)

• COPVs
  • Transport gasses under high pressure
  • Metal Liner
  • Wrapped by a Series of Carbon Strands

• Research Question: **Reliability of COPVs at Use Conditions for the Expected Mission Life**
  • Primary Focus on Strands
  • Secondary Focus on Relationship to Vessels
  • Strands Less Expensive to Test

• [https://www.nasa.gov/offices/nesc/home/Feature_COPVs_Jan-2012.html](https://www.nasa.gov/offices/nesc/home/Feature_COPVs_Jan-2012.html)
Analysis: Ordinary Least Squares
Example: Jet Turbine Engine Thrust

• A researcher is tasked with creating a model to explain jet turbine engine thrust* with the following predictors
  • $x_1$: primary speed of rotation
  • $x_2$: fuel flow rate
  • $x_3$: exhaust temperature
  • $x_4$: ambient temperature at time of test

• Begin analysis with ordinary least squares (OLS) regression

*Jet Turbine data in Table B.13, page 566 of Montgomery, Peck, and Vining (2012)
Checking Assumptions

• Proper OLS regression requires checking the basic assumptions of the modeling technique
• These assumptions can be explored through the residuals of the analysis
• Basic Assumptions
  1. constant error variance
  2. independent data
  3. roughly normal distribution
Normal Probability Plot: Jet Turbine Data

- Adequate fit to the data to assume a normal distribution
Residuals vs. Fits Plot: Jet Turbine Data

• Goal: To check for any special patterns
• Is the variance constant?
• No major issues in plot to suggest transforming data
Residuals vs. Explanatory Variables

• Check for unusual patterns. Not major issues in these plots.
Analysis: Stress Rupture Data
Description of Stress Rupture Test

• Stress Rupture
  • Failures occur after a period of time where there is no increase in load

• Failures are needed to determine reliability

• Must extrapolate from where test is performed versus where reliability predictions are made

• Test strands at higher loads and then extrapolate

• Need a model to make predictions

<table>
<thead>
<tr>
<th>Load</th>
<th>Time</th>
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<tbody>
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End of Test
(Survivors are right censored)
Basic Weibull Distribution

- Probability Density Function
  \[ f(t, \beta, \eta) = \frac{\beta}{\eta} \left( \frac{t}{\eta} \right)^{\beta - 1} e^{-\left( \frac{t}{\eta} \right)^{\beta}} \]

- Survivor Function = 1 – F(t)
  \[ S(t) = e^{-\left( \frac{t}{\eta} \right)^{\beta}} \]

\( \beta \): shape parameter
\( \eta \): scale parameter (characteristic life, time where 63.2% of units will fail)
Smallest Extreme Value Distribution

• Smallest extreme value (SEV) distribution is an alternate parameterization of the Weibull distribution

• SEV represents Weibull as a log-location-scale distribution
  • If $t_i$ is Weibull, then $\log(t_i)$ is SEV

• Weibull/SEV relationship mimics the Normal/Lognormal relationship

• Parameters
  • Log-location: $\mu = \log(\eta)$  Scale: $\sigma = \frac{1}{\beta}$

• Residuals
  • $e_i = \log t_i - \mu$
  • Scaled: $z_i = \beta e_i = \beta (\log t_i - \mu) = \frac{\log t_i - \mu}{\sigma}$

• Survivor Function
  • $S(t_i) = P(T > t_i) = e^{-e^{z_i}}$
Classic Stress Rupture Model: Weibull

• Classic Weibull Survival Function

\[ S(t_i) = P(T > t_i) = e^{-\left(\frac{t_i}{t_{ref}SR^\rho}\right)^\beta} \]

Note: \( \eta = t_{ref}SR^{-\rho} \)

• Observed Life Time: \( t_i \)

• \( SR \): Stress Ratio, ratio of stress level to strength scale parameter

• Critical Parameters:
  • \( \rho \): controls the relationship between the failure time and stress ratio (\( SR \))
  • \( \beta \): Shape parameter for time to Failure
  • \( t_{ref} \): Reference time to Failure
Classic Stress Rupture Model: SEV

• SEV Survival Function

\[ S(t_i) = e^{-\left(\frac{t_i}{t_{ref}SR^\rho}\right)^\beta} = e^{-e^{\beta(\log t_i - \theta + \rho \ln(SR))}} \]

where \( \theta = \log(t_{ref}) \) and \( \mu = \log(\eta) = \theta - \rho \ln(SR) \)

• Scaled Residuals

\[ z_i = \beta e_i = \beta (\log t_i - \mu) = \beta (\log t_i - \theta + \rho \ln(SR)) \]

• Used for predictions of the log probability for specific observations

Now working with a linear model, similar to simple linear regression
Weibull Regression: Fitting the Model
Stress Rupture Data

Stress Ratio versus Time on Hold

Right Censored Data
Stress Rupture Data with Ramp Failures

- Statisticians on team suggest a conditional analysis using only data that makes it to stress rupture hold.
- All ramp failures are dropped.
- In the literature, a procedure termed “effective time” has been suggested that incorporate ramp data as stress rupture data.
- Recent work has shown the errors and limitations of this “effective time” approach.

Note: X-Axis is Time on Hold

Failures on Ramp Have No Time On Hold!
True Structure of the Stress Rupture Model

\[ \mu = \log(\eta) = \theta - \rho \ln(SR) \]

- \( \rho \): controls the relationship between the failure time and stress ratio (SR)
- \( \beta \): Shape parameter for time to Failure
- \( t_{ref} \): Reference time to Failure

Stress Rupture model explains the behavior of the items on hold.

- Weibull regression gives us estimates for \( \rho, \beta \) and \( t_{ref} \)
Weibull Regression: Residuals
Weibull Residuals

• The Maximum Likelihood Estimates of the Model Depends upon:

\[ e_i = \log t_i - \mu_i \]

\[ = \log t_i - \log t_{ref} + \rho \ln SR_i \]

• \( e_i \) is the “Raw Residual”
• Equally Important is the “Scaled Residual,” \( \beta e_i \).
• These Residuals Contain All of the Relevant Information on \( t_{ref}, \rho, \beta \).
Proper Basis for Constructing Probability Plots

• Estimate the Model
• Construct the Scaled Residuals
• Calculate the Median Ranks for these Residuals (Overall not by SR!)
• Plot $\ln[-\ln(1 - mr_i)]$ versus the $\beta e_i$
• Method Extends Easily to More Complicated Models
Analogs to Standard Regression Residual Plots

• Proper Standard Regression Residual Plots Use Standardized Residuals
  • Let $y_i$ Represent the Observed Value for the $i^{th}$ Response
  • Let $\hat{y}_i$ Represent the Predicted Value for $y_i$ Based on the Assumed Model

• Raw Residual:
  \[ e_i = y_i - \hat{y}_i \]

• Let $s_i$ Be an Appropriate Estimate of the Standard Error for $e_i$. 
Standardized Regression Residual:

$$t_i = \frac{e_i}{s_i}$$

- Follows a true $t$-Distribution
- Standard Error Properly Accounts for the Variability in the Raw Residual
- Proper Basis for Tentative Outlier Detection: $\pm 3$
- Can Still Flag More Outliers than Are “Real”
- Issue: Multiple Comparisons

Proper Basis for Evaluating Standard Regression Model Adequacy.

Routinely Provided by Standard Software
Analogs to Standard Regression Residual Plots

• Extension Requires a New Residual: The “Probability Residual”
• Builds Off the Standard Probability Plot
• Let $r_{p,i}$ denote the “probability plot residual” defined by

$$r_{p,i} = \ln[-\ln(1 - mr_i)] - \beta e_i$$

$$mr_i = \hat{F}(t_i) = \frac{AR_i - 0.3}{n + 0.4}$$

where $AR_i$ is the adjusted rank and $n$ is the sample size.
Analogs to Standard Regression Residual Plots

• Standardized Regression Residual:
  \[ r_i = \frac{r_{p,i}}{s_i} \]

• Asymptotically, the \( r_i \):
  • Follow a Standard Normal Distribution
  • Need many failures to see the Asymptotic Behavior
  • Using \( \pm 3 \) as Cut-Off is Even More Prone to Indicate Outliers

• The Analog Plots Are the Best Basis for Evaluating Model Adequacy.
Standard Probability Plot

Weibull Probability Plot
Probability Residuals versus Scaled
Probability Residuals versus Serial Number
Probability Residuals versus Stress Ratio

![Probability Residuals vs. Stress Ratio Graph](image)
Nines in Reliability

• A statistic in reliability that is just another way of reporting the reliability of a product, system, etc.
• Literally just a count of the number of nines for reliability
• Ex: 99% reliability is 2 nines of reliability
  • Interpretation: One in 100
• 99,999% is 5 nines of reliability
  • Interpretation: One in 100,000

• Nines Calculation: \(-\log_{10}(1 - F(t))\)
Nines versus Log Time

Note: Slope of Each Prediction Line Is $-\beta$
Prediction Line Is $-\beta e_i$
(Negative Scaled Residual)